

INSTRUMENTATION ENGINEERING

SIGNALS AND SYSTEMS



Comprehensive Theory
with Solved Examples and Practice Questions





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Signals and Systems

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EDITIONS

First Edition : 2015
Second Edition : 2016
Third Edition : 2017
Fourth Edition : 2018
Fifth Edition : 2019
Sixth Edition : 2020
Seventh Edition : 2021
Eighth Edition : 2022
Ninth Edition : 2023

Tenth Edition : 2024

CONTENTS

Signals and Systems

CHAPTER 1

Introduction to Signals 2-54

1.1	Introduction.....	2
1.2	Elementary Signals.....	2
1.3	Classification of Signals	17
1.4	Basic Operations on Signals.....	37
	<i>Objective Brain Teasers</i>	50
	<i>Conventional Brain Teasers</i>	54

CHAPTER 2

Introduction to Systems 55-96

2.1	Introduction.....	55
2.2	Continuous-time and discrete-time systems.....	56
2.3	Classification of Systems	56
2.4	Linear Time-Invariant (LTI) Systems.....	64
2.5	Continuous time LTI systems	65
2.6	Discrete-time LTI Systems	78
2.7	LTI System Properties and the Impulse Response	82
2.8	Step Response of an LTI System	85
	<i>Objective Brain Teasers</i>	88
	<i>Conventional Brain Teasers</i>	93

CHAPTER 3

Continuous-time Fourier Series 97-123

3.1	Introduction.....	97
3.2	Different Forms of Fourier Series.....	97
3.3	Symmetry Conditions in Fourier Series	101
3.4	Dirichlet Conditions.....	104
3.5	Properties of Fourier Series	106
3.6	Systems with Periodic Inputs.....	113
3.7	Limitations of Fourier Series	113
	<i>Objective Brain Teasers</i>	114
	<i>Conventional Brain Teasers</i>	118

CHAPTER 4

Continuous Time Fourier Transform 124-164

4.1	Introduction.....	124
4.2	The Definition.....	124
4.3	Fourier Transform of Some Basic Signals.....	125
4.4	Inverse Fourier Transform of Some Basic Functions....	129
4.5	Properties of Fourier Transform	133
4.6	Fourier Transform of Periodic Signal.....	149
4.7	Application of Fourier Transform	151
4.8	Ideal and Practical Filters.....	152
4.9	Energy Spectral Density (ESD).....	153
4.10	Power Spectral Density (PSD).....	154
4.11	Correlation.....	155
4.12	Limitation of Fourier Transform and its Solution....	157
	<i>Objective Brain Teasers</i>	158
	<i>Conventional Brain Teasers</i>	162

CHAPTER 5

Laplace Transform 165-210

5.1	Introduction.....	165
5.2	The Definition.....	165
5.3	Relationship between Laplace Transform and Fourier Transform	166
5.4	Eigen Value and Eigen Function.....	166
5.5	Region of Convergence (ROC) for Laplace Transform... 166	
5.6	Laplace Transforms to Some Basic Signals	168
5.7	Properties of Laplace Transform.....	175
5.8	Inverse Laplace Transform	182
5.9	LTI System and Laplace Transform.....	186
5.10	Interconnection of LTI Systems (Block Diagrams) ... 191	
5.11	Laplace Transform of Causal Periodic Signals.....	192
5.12	Unilateral Laplace Transform	193
5.13	Properties of Unilateral Laplace Transform (ULT) ... 195	
5.14	Application of Laplace Transform in Solving Differential Equations.....	199
	<i>Objective Brain Teasers</i>	203
	<i>Conventional Brain Teasers</i>	209

CHAPTER 6**Sampling..... 211-228**

6.1	Introduction.....	211
6.2	The Sampling Theorem	211
6.3	Sampling Techniques	215
6.4	Sampling Theorem for Band Pass Signals.....	217
6.5	Reconstruction of Signal.....	219
	<i>Objective Brain Teasers</i>	224
	<i>Conventional Brain Teasers</i>	227

CHAPTER 7**z-Transform.....229-279**

7.1	Introduction.....	229
7.2	The Definition.....	230
7.3	Region of Convergence for z-transform	230
7.4	z-Transform of Some Basic Signals	233
7.5	Properties of z-Transform.....	240
7.6	Inverse z-Transform	248
7.7	Discrete-time LTI Systems and z-Transform.....	254
7.8	z-Transform of Causal Periodic Signals.....	260
7.9	Relation between Laplace Transform and z-Transform.....	260
7.10	Unilateral z-Transform	262
7.11	Properties of Unilateral z-transform (UZT)	263
7.12	z-Transform Solution of Linear Difference Equations... <i>Objective Brain Teasers</i>	266 269
	<i>Conventional Brain Teasers</i>	274

CHAPTER 8**Fourier Analysis of Discrete Time Signals...280-314**

8.1	Introduction to Discrete Time Fourier Series (DTFS) ...	280
8.2	The Definition.....	280
8.3	Properties of DTFS	282
8.4	Introduction to Discrete Time Fourier Transform ...	282
8.5	The Definition: DTFT	282
8.6	DTFT of some Basic Signals	284
8.7	Properties of DTFT	288
8.8	Fourier Transform Pairs Using Inverse DTFT	296
8.9	Fourier Transform of Periodic Signals	298

8.10	LTI System Analysis and DTFT.....	299
8.11	Application of DTFT	300
8.12	Ideal and Practical Filters.....	302
8.13	Relationship between CTFT and DTFT	305
8.14	Energy Spectral Density	306
8.15	Power Spectral Density: (PSD).....	306
8.16	Correlation.....	307
	<i>Objective Brain Teasers</i>	307
	<i>Conventional Brain Teasers</i>	311

CHAPTER 9**Discrete Fourier Transform (DFT).....315-332**

9.1	Introduction.....	315
9.2	The Definition.....	316
9.3	Properties of DFT	320
9.4	Introduction to FFT (Fast Fourier Transform)	326
	<i>Objective Brain Teasers</i>	326
	<i>Conventional Brain Teasers</i>	330

CHAPTER 10**Digital Filters.....333-387**

10.1	Introduction.....	333
10.2	Filter Basics.....	333
10.3	Butterworth Filters	334
10.4	Digital Filters.....	335
10.5	Basics Structures for IIR Systems	336
10.6	Basic Structures for FIR Systems	347
10.7	IIR Filter Design from Continuous-Time Filters.....	351
10.8	Impulse Invariant Method	351
10.9	Design of IIR Filter by Approximation of Derivatives ...	357
10.10	IIR Filter Design by the Bilinear Transformation.....	360
10.11	Design of FIR Filters.....	364
10.12	Design of Linear Phase FIR Filters using Frequency Sampling Method.....	374
10.13	Lattice Structure of FIR Filter.....	376
10.14	Comparison of Designing Methods.....	383
10.15	Comparison between FIR and IIR Filter.....	384
	<i>Objective Brain Teasers</i>	384
	<i>Conventional Brain Teasers</i>	386

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Signals and Systems

INTRODUCTION TO SIGNALS AND SYSTEMS

This book starts with basic and extensive chapter on signals in which continuous and discrete-time case are discussed in parallel. A variety of basic signals, functions with their mathematical description, representation and properties are incorporated. A substantial amount of examples are given for quick sketching of functions. A chapter on systems is discussed separately which deals with classification of systems, both in continuous and discrete domain and more emphasize is given to LTI systems and analytical as well as graphical approach is used to understand convolution operation. These two chapters makes backbone of the subject.

Further we shall proceed to transform calculus which is important tool of signal processing. A logical and comprehensive approach is used in sequence of chapters. The continuous time Fourier series which is base to the Fourier transform, deals with periodic signal representation in terms of linear complex exponential, is discussed.

The Fourier transform is discussed before Laplace transform. The sampling, a bridge between continuous-time and discrete-time, is discussed to understand discrete-time domain.

A major emphasis is given on proof of the properties so that students can understand and analyzes fundamental easily.

A point wise recapitulation of all the important points and results in every chapter proves helpful to students in summing up essential developments in the chapter which is an integral part of any competitive examination.

Introduction to Signals

1.1 INTRODUCTION

A signal is any quantity having information associated with it. It may also be defined as a function of one or more independent variables which contain some information.

The function defines mapping from one set to another and similarly a signal may also be defined as mapping from one set (domain) to another (range). e.g.

- A speech signal would be represented by acoustic pressure as a function of time.
- A monochromatic picture would be represented by brightness as a function of two spatial variables.
- A voltage signal is defined by a voltage across two points varying as function of time.
- A video signal, in which color and intensity as a function of 2-dimensional space (2D) and 1-dimensional time (i.e. hybrid variables).



In this course of “signals and systems”, we shall focus on signals having only one variable and will consider ‘time’ as independent variable.

1.2 ELEMENTARY SIGNALS

These signals serve as basic building blocks for construction of somewhat more complex signals. The list of elementary signals mainly contains singularity functions and exponential functions.

These elementary signals are also known as basic signals/standard signals.

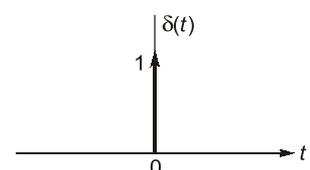
Let us discuss these basic signals one-by-one.

1.2.1 Unit Impulse Function

A continuous-time unit impulse function $\delta(t)$, also called as Dirac delta function is defined as

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t) dt = 1$$

The unit-impulse function is represented by an arrow with strength of ‘1’ which represents its ‘area’ or ‘weight’.



The above definition of an impulse function is more generalised and can be represented as limiting process without any regard to shape of a pulse. For example, one may define impulse function as a limiting case of rectangular pulse, triangular pulse Gaussian pulse, exponential pulse and sampling pulse as shown below:

Sl. No.	Type of Impulse	Graph
1.	Rectangular Pulse $\delta(t) = \lim_{\epsilon \rightarrow 0} p(t)$	
2.	Triangular Pulse $\delta(t) = \begin{cases} \lim_{\tau \rightarrow 0} \frac{1}{\tau} \left[1 - \frac{ t }{\tau} \right] & ; t < \tau \\ 0 & ; t > \tau \end{cases}$	
3.	Gaussian Pulse $\delta(t) = \lim_{\tau \rightarrow 0} \frac{1}{\tau} \left[e^{-t^2/\tau^2} \right]$	
4.	Exponential Pulse $\delta(t) = \lim_{\tau \rightarrow 0} \frac{1}{2\tau} \left[e^{- t /\tau} \right]$	
5.	Sampling Function $\int_{-\infty}^{\infty} \frac{k}{\pi} \text{Sa}(kt) dt = 1$	

Properties of Continuous Time Unit Impulse Function

(i) Scaling property:

$$\delta(at) = \frac{1}{|a|} \delta(t) \quad ; \quad 'a' \text{ is a constant, positive or negative}$$

Proof:

$$\delta(at) = \frac{1}{|a|} \delta(t)$$

Integrating above equation on both the sides with respect to 't'.

$$\int_{-\infty}^{+\infty} \delta(at) dt = \int_{-\infty}^{+\infty} \frac{1}{|a|} \delta(t) dt$$

Let

$$at = \tau$$

$$a \cdot dt = d\tau \quad ; \quad 'a' \text{ is a constant, positive or negative} \quad \text{or} \quad |a| \cdot dt = d\tau$$

Now,

$$\int_{-\infty}^{+\infty} \delta(at) dt = \int_{-\infty}^{+\infty} \delta(\tau) \cdot \frac{d\tau}{|a|} = \int_{-\infty}^{+\infty} \frac{1}{|a|} \delta(\tau) \cdot d\tau \quad \text{By definition, } \int_{-\infty}^{+\infty} \delta(t) dt = \int_{-\infty}^{+\infty} \delta(\tau) d\tau = 1$$

(ii) Product property/multiplication property:

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$$

Proof:

The function $\delta(t - t_0)$ exists only at $t = t_0$. Let the signal $x(t)$ be continuous at $t = t_0$.

Therefore,
$$x(t)\delta(t - t_0) = x(t)|_{t=t_0} \cdot \delta(t - t_0) = x(t_0)\delta(t - t_0)$$



Important Expressions

- $\delta(at \pm b) = \frac{1}{|a|} \delta\left(t \pm \frac{b}{a}\right)$
- $\delta(-t) = \delta(t) \quad \because \delta(t) \text{ is an even function of time.}$
- $x(t)\delta(t) = x(0)\delta(t)$
- $\int_{-\infty}^{+\infty} x(t)\delta(t) dt = x(0)$

(iii) Sampling property:

$$\int_{-\infty}^{+\infty} x(t)\delta(t - t_0) dt = x(t_0)$$

Proof:

Using product property of impulse function

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$$

Integrating above equation on both the sides with respect to 't'.

$$\int_{-\infty}^{+\infty} x(t)\delta(t - t_0) dt = \int_{-\infty}^{+\infty} x(t_0)\delta(t - t_0) dt = x(t_0) \int_{-\infty}^{+\infty} \delta(t - t_0) dt = x(t_0)$$

$$\int_{-\infty}^{+\infty} x(t)\delta(t) dt = x(0)$$

(iv) The first derivative of unit step function results in unit impulse function.

$$\delta(t) = \frac{d}{dt}u(t)$$

Proof:

Let the signal $x(t)$ be continuous at $t = 0$.

$$\begin{aligned} \text{Consider the integral } \int_{-\infty}^{+\infty} \frac{d}{dt}[u(t)] x(t) dt &= [u(t)x(t)]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} x'(t) u(t) dt \\ &= x(\infty) - \int_0^{\infty} x'(t) dt = x(\infty) - [x(t)]_0^{\infty} = x(0) \end{aligned} \quad \dots(i)$$

$$\text{We know from sampling property } x(0) = \int_{-\infty}^{+\infty} x(t) \delta(t) dt \quad \dots(ii)$$

From equations (i) and (ii), we get

$$\int_{-\infty}^{+\infty} \frac{d}{dt}[u(t)] x(t) dt = \int_{-\infty}^{+\infty} x(t) \delta(t) dt ; \quad \text{On comparing, we get } \delta(t) = \frac{d}{dt}u(t)$$

(v) Derivative property:

$$\int_{t_1}^{t_2} x(t) \delta^n(t - t_0) dt = (-1)^n x^n(t) \Big|_{t=t_0} ; \quad t_1 < t_0 < t_2 \text{ and suffix } n \text{ means } n^{\text{th}} \text{ derivative}$$

$$\text{where, } \delta^n(t - t_0) = \frac{d^n}{dx^n} \delta(t)$$

Proof:

Let the signal $x(t)$ be continuous at $t = t_0$ where $t_1 < t_0 < t_2$.

$$\text{Consider the derivative } \frac{d}{dt}[x(t) \delta(t - t_0)] = x(t) \delta'(t - t_0) + x'(t) \delta(t - t_0)$$

Integrating above equation on both the sides with respect to 't'.

$$\begin{aligned} \int_{t_1}^{t_2} \frac{d}{dt}[x(t) \delta(t - t_0)] dt &= \int_{t_1}^{t_2} x(t) \delta'(t - t_0) dt + \int_{t_1}^{t_2} x'(t) \delta(t - t_0) dt \\ [x(t) \delta(t - t_0)]_{t_1}^{t_2} &= \int_{t_1}^{t_2} x(t) \delta'(t - t_0) dt + \int_{t_1}^{t_2} x'(t) \delta(t - t_0) dt \\ [x(t_2) \delta(t_2 - t_0) - x(t_1) \delta(t_1 - t_0)] &= \int_{t_1}^{t_2} x(t) \delta'(t - t_0) dt + \int_{t_1}^{t_2} x'(t) \delta(t - t_0) dt \end{aligned}$$

Here, $\delta(t_1 - t_0) = 0$ and $\delta(t_2 - t_0) = 0$ because $t_0 \neq t_1$ or $t_0 \neq t_2$

$$\text{So, } 0 = \int_{t_1}^{t_2} x(t) \delta'(t - t_0) dt + \int_{t_1}^{t_2} x'(t) \delta(t - t_0) dt$$

$$\begin{aligned} \int_{t_1}^{t_2} x(t) \delta'(t - t_0) dt &= (-1) \int_{t_1}^{t_2} x'(t) \delta(t - t_0) dt \quad (\because \text{using sampling property}) \\ \Rightarrow &= (-1) x'(t_0) \end{aligned}$$

Hence,
$$\int_{t_1}^{t_2} x(t) \delta'(t - t_0) dt = (-1)^1 x'(t_0)$$

If same procedure is repeated for second derivative, we get

$$\int_{t_1}^{t_2} x(t) \delta''(t - t_0) dt = (-1)^2 x''(t_0)$$

On generalising aforementioned results, we get

$$\int_{t_1}^{t_2} x(t) \delta^n(t - t_0) dt = (-1)^n x^n(t_0)$$

(vi) Shifting Property:

According to shifting property, any signal can be produced as combination of weighted and shifted impulses.

$$x(t) = \int_{-\infty}^{+\infty} x(\tau) \delta(t - \tau) d\tau$$

Proof:

Using product property, $x(t) \delta(t - t_0) = x(t_0) \delta(t - t_0)$

Replacing t_0 by τ , $x(t) \delta(t - \tau) = x(\tau) \delta(t - \tau)$

Integrating above equation on both the sides with respect to ' τ '.

$$\int_{-\infty}^{+\infty} x(t) \delta(t - \tau) d\tau = \int_{-\infty}^{+\infty} x(\tau) \delta(t - \tau) d\tau ; \quad x(t) \int_{-\infty}^{+\infty} \delta(t - \tau) d\tau = \int_{-\infty}^{+\infty} x(\tau) \delta(t - \tau) d\tau$$

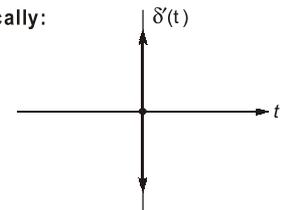
$$x(t) \cdot 1 = \int_{-\infty}^{+\infty} x(\tau) \delta(t - \tau) d\tau ; \quad x(t) = \int_{-\infty}^{+\infty} x(\tau) \delta(t - \tau) d\tau$$

(vii) The derivative of impulse function is known as **doublet** function.

$$\delta'(t) = \frac{d}{dt} \delta(t)$$

Area under the **doublet** function is always zero.

Graphically:

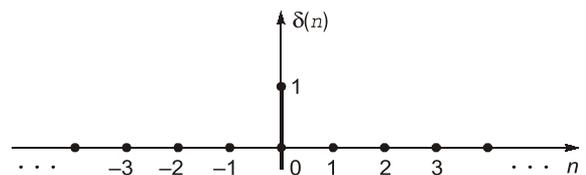


Discrete-Time Case

The discrete time unit impulse function $\delta[n]$, also called unit sample sequence or delta sequence is defined as

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

It is also known as **Kronecker delta**.



Properties of Discrete Time Unit Impulse Sequence

(i) Scaling property:

$$\delta[kn] = \delta[n]; k \text{ is an integer}$$

Proof:

By definition of unit impulse sequence

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Similarly,

$$\delta[kn] = \begin{cases} 1, & kn = 0 \\ 0, & kn \neq 0 \end{cases} = \begin{cases} 1, & n = \frac{0}{k} = 0 \\ 0, & n \neq \frac{0}{k} \neq 0 \end{cases} = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} = \delta[n]$$

(ii) Product property:

$$x[n] \delta[n - n_0] = x[n_0] \delta[n - n_0]$$

From definition,
$$\delta[n - n_0] = \begin{cases} 1, & n = n_0 \\ 0, & n \neq n_0 \end{cases}$$

We see that impulse has a non zero value only at $n = n_0$

Therefore,
$$x[n] \delta[n - n_0] = x[n]_{n=n_0} \delta[n - n_0]$$

$$x[n] \delta[n - n_0] = x[n_0] \delta[n - n_0]$$

(iii) Shifting property:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k] \delta[n - k]$$

Proof:

From product property,
$$x[n] \delta[n - n_0] = x[n_0] \delta[n - n_0]$$

Replacing n_0 by ' k '
$$x[n] \delta[n - k] = x[k] \delta[n - k]$$

$$\Rightarrow \sum_{k=-\infty}^{+\infty} x[n] \delta[n - k] = \sum_{k=-\infty}^{+\infty} x[k] \delta[n - k] \quad \Rightarrow x[n] \sum_{k=-\infty}^{+\infty} \delta[n - k] = \sum_{k=-\infty}^{+\infty} x[k] \delta[n - k]$$

$$\Rightarrow x[n] \cdot 1 = \sum_{k=-\infty}^{+\infty} x[k] \delta[n - k] ; \quad x[n] = \sum_{k=-\infty}^{+\infty} x[k] \delta[n - k]$$

(iv) The first difference of unit step sequence results in unit impulse sequence.

$$\delta[n] = u[n] - u[n - 1]$$

Proof:

By definition of unit step sequence

$$\begin{aligned} u[n] &= \sum_{k=0}^{\infty} \delta[n - k] && \dots(i) \\ &= \delta[n] + \sum_{k=1}^{\infty} \delta[n - k] \end{aligned}$$

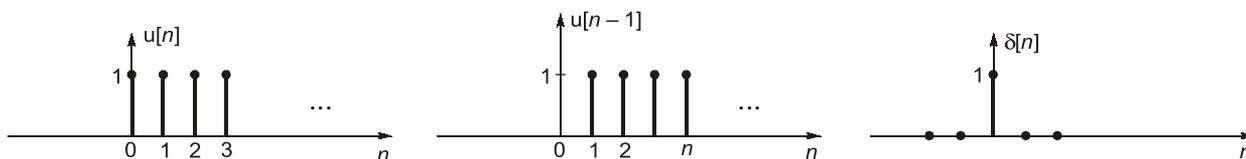
But,

$$u[n - 1] = \sum_{k=1}^{\infty} \delta[n - k]$$

We get,
Graphically we can see,

$$u[n] = \delta[n] + u[n-1]$$

$$\delta[n] = u[n] - u[n-1]$$



Summary Table:

S.No.	Properties of CT unit Impulse Function	Properties of DT unit impulse sequence
1.	$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & \text{otherwise} \end{cases}$ and $\int_{-\infty}^{\infty} \delta(t) dt = 1$	$\delta[n] = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{otherwise} \end{cases}$
2.	$x(t) \delta(t - t_0) = x(t_0) \delta(t - t_0)$	$x[n] \delta[n - k] = x[k] \delta[n - k]$
3.	$\delta(t) = \frac{d}{dt} u(t)$	$\delta[n] = u[n] - u[n-1]$
4.	$\int_0^{\infty} \delta(t - \tau) d\tau = u(t)$	$\sum_{k=0}^{\infty} \delta[n - k] = u[n]$
5.	$x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau$	$x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - k]$
6.	$\int_{-\infty}^{\infty} x(t) \delta(t - t_0) dt = x(t_0)$	$\sum_{n=-\infty}^{\infty} x[n] \delta[n - n_0] = x[n_0]$
7.	$\delta(at) = \frac{1}{ a } \delta(t)$ $\delta(at \pm b) = \frac{1}{ a } \delta\left(t \pm \frac{b}{a}\right)$ $\delta(-t) = \delta(t)$	$\delta[kn] = \delta[n]$ $\delta[-n] = \delta[n]$
8.	$\int_{t_1}^{t_2} x(t) \delta(t) dt = \begin{cases} x(0), & t_1 < t < t_2 \\ 0, & \text{otherwise} \end{cases}$	
9.	$\int_{t_1}^{t_2} x(t) \delta^n(t - t_0) dt = (-1)^n x^n(t_0), t_1 < t_0 < t_2$ where suffix n mean n^{th} derivative	
10.	$\delta'(t) = \frac{d}{dt} \delta(t)$	

EXAMPLE : 1.1

The Dirac delta function $\delta(t)$ is defined as

$$\begin{aligned} \text{(a)} \quad \delta(t) &= \begin{cases} 1 & ; \quad t = 0 \\ 0 & ; \quad \text{otherwise} \end{cases} & \text{(b)} \quad \delta(t) &= \begin{cases} 1 & ; \quad t = 0 \\ 0 & ; \quad \text{otherwise} \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t) dt = 1 \\ \text{(c)} \quad \delta(t) &= \begin{cases} \infty & ; \quad t = 0 \\ 0 & ; \quad \text{otherwise} \end{cases} & \text{(d)} \quad \delta(t) &= \begin{cases} \infty & ; \quad t = 0 \\ 0 & ; \quad \text{otherwise} \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t) dt = 1 \end{aligned}$$

Solution : (d)

EXAMPLE : 1.2

The integral $\int_{-\infty}^{\infty} \delta\left(t - \frac{\pi}{6}\right) 6 \sin(t) dt$ evaluate to

$$\begin{aligned} \text{(a)} \quad 6 & & \text{(b)} \quad 3 \\ \text{(c)} \quad 1.5 & & \text{(d)} \quad 0 \end{aligned}$$

Solution : (b)

Given signal is
$$x(t) = \int_{-\infty}^{\infty} \delta\left(t - \frac{\pi}{6}\right) 6 \sin t dt$$

By applying shifting property of unit impulse function

$$\begin{aligned} \int_{-\infty}^{\infty} x(\lambda) \delta(t - \lambda) d\lambda &= x(t) \\ \int_{-\infty}^{\infty} \delta\left(t - \frac{\pi}{6}\right) 6 \sin(t) dt &= 6 \cdot \sin \frac{\pi}{6} = 6 \times \frac{1}{2} = 3 \end{aligned}$$

EXAMPLE : 1.3

If $y(t) + \int_0^{\infty} y(\tau) x(t - \tau) d\tau = \delta(t) + x(t)$, then $y(t)$ is

$$\begin{aligned} \text{(a)} \quad u(t) & & \text{(b)} \quad \delta(t) \\ \text{(c)} \quad r(t) & & \text{(d)} \quad 1 \end{aligned}$$

Solution : (b)

As we know that
$$\int_0^{\infty} y(\lambda) x(t - \lambda) d\lambda = x(t)$$

So, $y(t) = \delta(t)$ satisfies the given equation

EXAMPLE : 1.4

Which of the following is NOT a property of impulse function?

$$\begin{aligned} \text{(a)} \quad x(t) \delta(t - t_0) &= x(t_0) \delta(t - t_0) & \text{(b)} \quad x(t) * \delta(t - t_0) &= x(t - t_0) \\ \text{(c)} \quad \int_{t_1}^{t_2} x(t) \delta(t - t_0) dt &= x(t_0); t_1 < t < t_2 & \text{(d)} \quad \int_{-\infty}^{+\infty} x(t) \frac{d^n}{dx^n} \delta(t - t_0) dt &= \frac{d^n}{dt^n} x(t) \Big|_{t=t_0} \end{aligned}$$

Solution : (d)

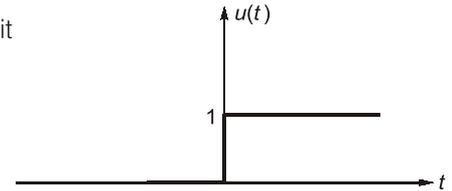
By derivative property
$$\int_{-\infty}^{+\infty} x(t) \frac{d^n}{dx^n} \delta(t - t_0) dt = (-1)^n x^n(t) \Big|_{t=t_0}$$

1.2.2 Unit Step Function

The continuous-time unit step function, also called “Heaviside” unit function, is defined as

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

The function value at $t = 0$ is indeterminate (discontinuous)



Properties of unit step function:

(i) The unit step function can be represented as integral of weighted, shifted impulses.

$$u(t) = \int_0^{\infty} \delta(t - \tau) d\tau$$

Proof:

According to the shifting property $x(t) = \int_{-\infty}^{+\infty} x(\tau) \delta(t - \tau) d\tau$ Let, $x(t) = u(t)$

$$u(t) = \int_{-\infty}^{+\infty} u(\tau) \delta(t - \tau) d\tau = \int_0^{+\infty} \delta(t - \tau) d\tau$$

Since,

$$u(\tau) = 0 ; -\infty < \tau < 0$$

$$u(\tau) = 1 ; \tau > 0$$

(ii) **Scaling property:**

$$u(at) = u(t), \quad a > 0$$

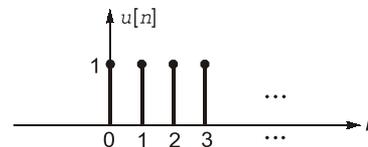


- The unit step function is continuous for all t , except for $t = 0$ where sudden change take place (i.e. discontinuity).
- $\langle u \rangle = \frac{1}{2}$ (The average value)

Discrete-Time Case

The discrete time unit-step sequence $u[n]$ is defined as,

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



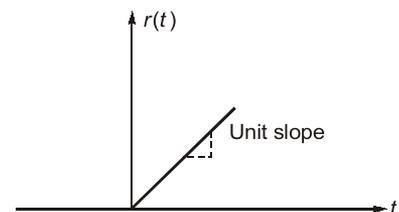
1.2.3 Unit Ramp Function:

A continuous time unit ramp function is defined as

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Also,

$$r(t) = tu(t)$$



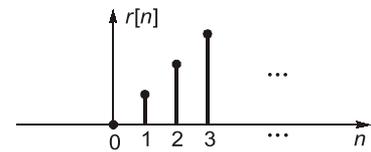
Discrete-Time Case

A discrete-time unit ramp sequence is defined as

$$r[n] = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Also,

$$r[n] = nu[n]$$



Important Expressions

- $r(t) = \int_{-\infty}^t u(\tau) d\tau$
- $r(t) = \int_{-\infty}^t \int_{-\infty}^{\alpha} \delta(\tau) d\tau d\alpha$
- $r[n] = nu[n]$

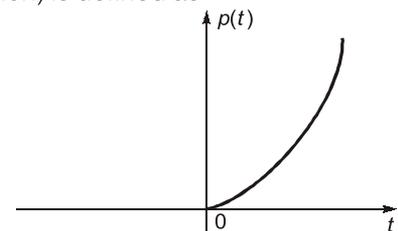
1.2.4 Unit Parabolic Function:

A continuous-time unit parabolic function $p(t)$ (unit acceleration function) is defined as

$$p(t) = \begin{cases} \frac{t^2}{2}, & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Also,

$$p(t) = \frac{tr(t)}{2} = \frac{t^2}{2}u(t)$$

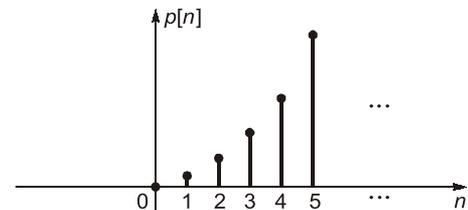


Discrete-Time Case

The discrete time unit parabolic sequence $p[n]$ is defined as

$$p[n] = \begin{cases} \frac{n^2}{2}, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Also, $p[n] = \frac{nr[n]}{2} = \frac{n^2u[n]}{2}$



1.2.5 Signum Function:

The continuous time signum function, $sgn(t)$ is defined as

$$sgn(t) = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \end{cases}$$

We see,

$$u(t) - u(-t) = sgn(t)$$

also

$$u(t) + u(-t) = 1$$

We get,

$$sgn(t) = 2u(t) - 1$$

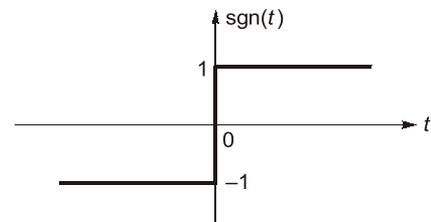
Keeping following facts in the mind that is

(i) $\lim_{\alpha \rightarrow 0} e^{-\alpha t} = 1, t > 0$

(ii) $\lim_{\alpha \rightarrow 0} e^{\alpha t} = 1, t < 0$

The positive half of signum function can be represented as $\lim_{\alpha \rightarrow 0} e^{-\alpha t}u(t)$ and the negative half, as

$$\lim_{\alpha \rightarrow 0} e^{\alpha t}u(-t).$$

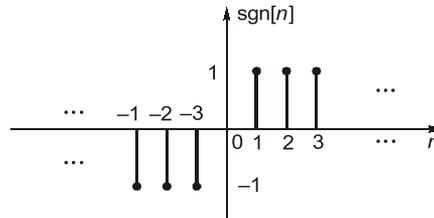


Mathematically, $\text{sgn}(t)$ can be represented as limiting case of exponential as

$$\text{sgn}(t) = \lim_{\alpha \rightarrow 0} [e^{-\alpha t} u(t) - e^{\alpha t} u(-t)]$$

The discrete-time signum sequence is defined

$$\text{sgn}[n] = \begin{cases} -1, & n < 0 \\ 0, & n = 0 \\ 1, & n > 0 \end{cases}$$



also,

$$\text{sgn}[n] = u[n-1] - u[-n-1]$$

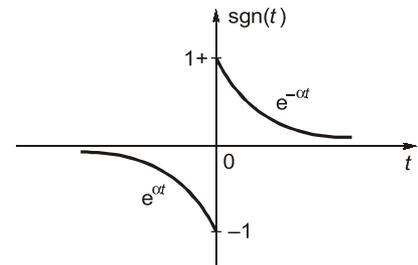


Figure: $\text{sgn}(t)$ as limiting curve of exponential function

1.2.6 Exponentials and Sinusoidal Signal

A general form of complex exponential signal is

$$x(t) = Ce^{\alpha t}$$

Depending upon values of C and α we further classify complex exponential as

- (i) Real exponential : Both C and α are real.
- (ii) Periodic complex exponential : C is real α is purely imaginary.
- (iii) Sinusoidal
- (iv) Complex exponential : Both C and α are complex.

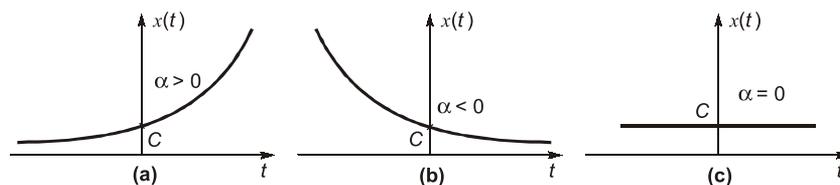
(i) Real Exponential Signal

Continuous-time case

A continuous-time real exponential signal, in general form can be defined as

$$x(t) = Ce^{\alpha t} ; \text{ both } C \text{ and } \alpha \text{ are real}$$

- For $\alpha > 0$, the signal is growing exponential signal.
- For $\alpha < 0$, the signal is decaying exponential signal.
- For $\alpha = 0$, $x(t)$ is constant signal.



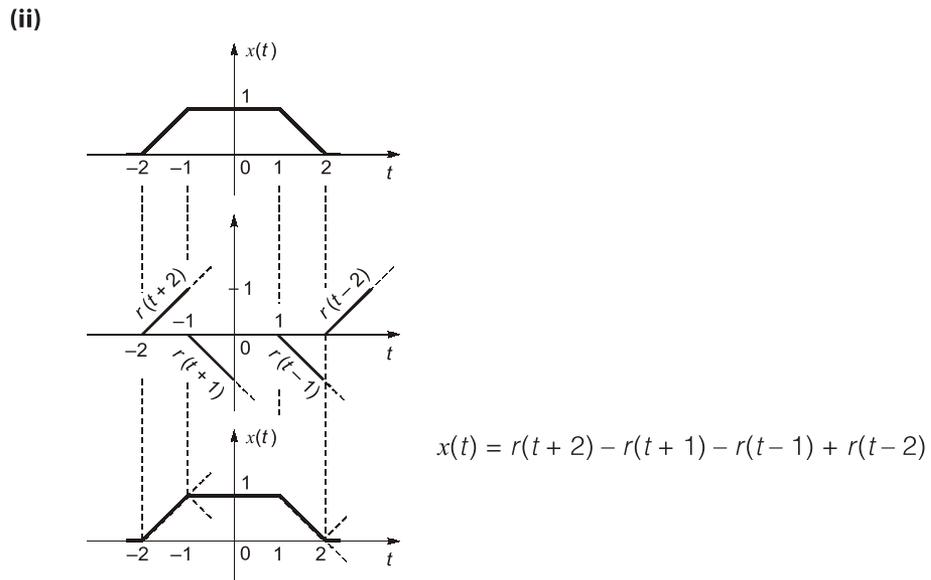
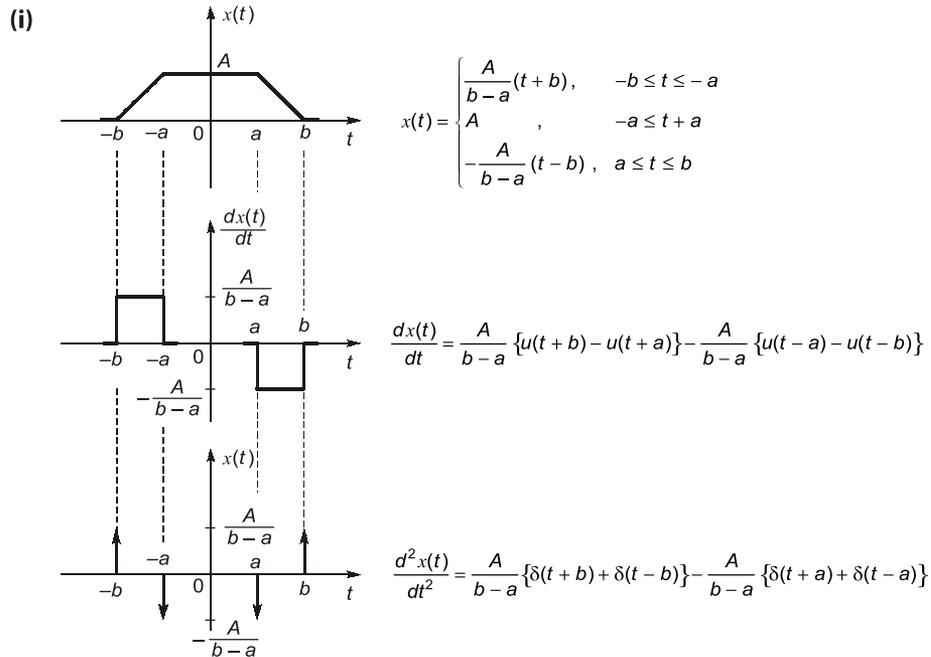
Discrete-time case

The discrete-time real exponential sequence, in general form can be defined as

$$x[n] = Ca^n, \text{ for all } n \quad \text{where } a = e^{\beta} \text{ and } \beta \text{ is real.}$$

Discrete-time real exponential are often used to describe population growth as a function of generation and investments as a function of month, or year.

Solution :



OBJECTIVE BRAIN TEASERS

- Q.1** The odd component of the signal $x(t) = e^{-2t} \cos t$ is
 (a) $\cosh(2t) \cos t$ (b) $-\sinh(2t) \cos t$
 (c) $-\cosh(2t) \cos t$ (d) $\sinh(2t) \cos t$

Q.2 A discrete time system is given as:

$$x[n] = \cos\left(\frac{n}{4}\right) \cdot \sin\left(\frac{\pi n}{4}\right)$$

The signal is

- (a) periodic with 8 (b) periodic with $8(\pi + 1)$
 (c) periodic with 4 (d) non-periodic

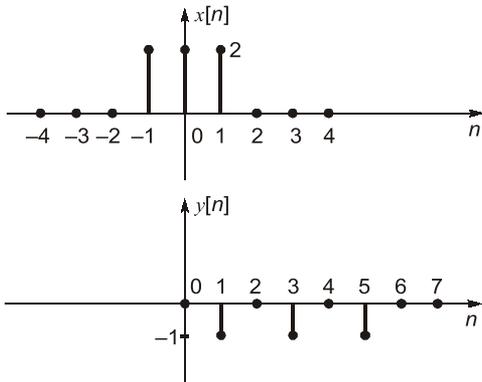
Q.3 The power of signal $x[n] = (-1)^n u[n]$ is _____ W.

Q.4 A discrete time signal is given as

$$x[n] = \cos\left(\frac{\pi n}{3}\right) \cdot (u[n] - u[n-6])$$

The energy of the signal is _____ J.

Q.5 Two functions $x[n]$ and $y[n]$ are shown in following figures.



If $y[n] = \alpha x\left[\frac{n-n_0}{k}\right]$ then value of $n_0 + \alpha + k$ is _____.

Q.6 Consider a discrete time signal as follows:

$$x[n] = \begin{cases} 1 & ; n=1 \\ -1 & ; n=-1 \\ 0 & ; \text{otherwise} \end{cases}$$

If $y[n] = x[n] + x[-n]$, then the energy of the signal $y[n]$ will be

- (a) 0 (b) 1
(c) 2 (d) 4

Q.7 A continuous time signal is defined as,

$$x(t) = 4\cos\left(\frac{2\pi}{3}t + 40^\circ\right) + 3\sin\left(\frac{4\pi}{5}t + 20^\circ\right).$$

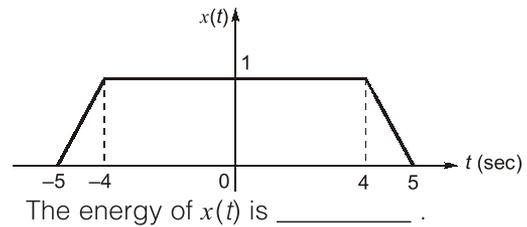
The fundamental time period of $x(t)$ is

- (a) 30π sec (b) 15π sec
(c) 15 sec (d) 30 sec

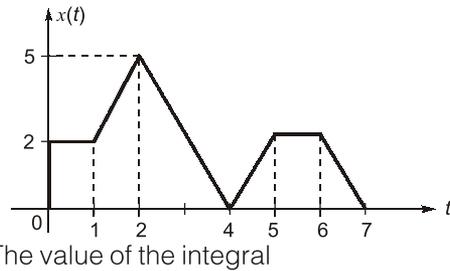
Q.8 The conjugate antisymmetric part of the sequence, $x(n) = [-5, -3 + 2j, 4j, 8 + 9j]$ is

- (a) $[-4 + 4.5j, -2.5 + 2j, -2j, -2.5 + 2j, 4 + 4.5j]$
(b) $[-4 + 4.5j, -2.5 + 2j, 2j, 2.5 + 2j, 4 + 4.5j]$
(c) $[-4 - 4.5j, -2.5 + 2j, -2j, 2.5 + 2j, 4 + 4.5j]$
(d) $[-4 - 4.5j, -2.5 + 2j, -2j, 2.5 - 2j, -4 + 4.5j]$

Q.9 Consider the trapezoidal pulse $x(t)$ shown in the figure below:



Q.10 A signal $x(t)$ is given by,

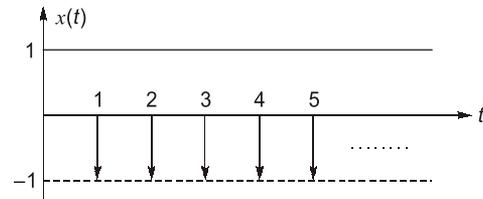


$$I = \int_{-\infty}^{\infty} x(-t+1)\delta'(t+2.5)dt \text{ is } \underline{\hspace{2cm}}.$$

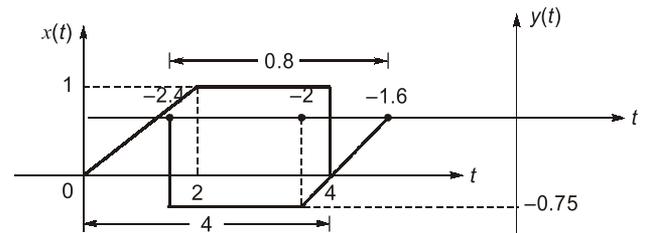
Q.11 For the signal $x(t)$ shown below, the value of

$$\int_{-\infty}^t x(\tau) d\tau \Big|_{t=0.5} \text{ is } \underline{\hspace{2cm}}.$$

(Upto two decimal places).



Q.12 Two signals $x(t)$ and $y(t)$ are shown below,



If $y(t) = ax(bt + c)$ then the value of $4a + b - c$ is _____.

ANSWERS KEY

1. (b) 2. (d) 3. (0.5) 4. (3) 5. (4.5)
6. (a) 7. (c) 8. (b) 9. (8.67)
10. (2.5) 11. (0.5) 12. (0)

HINTS & EXPLANATIONS

1. (b)

$$x(t) = e^{-2t} \cos t$$

$$x(-t) = e^{2t} \cos(-t) = e^{2t} \cos t$$

Odd component,

$$x_o(t) = \frac{1}{2} [x(t) - x(-t)]$$

$$x_o(t) = \cos t \left[\frac{e^{-2t} - e^{2t}}{2} \right] = -\cos t \left[\frac{e^{2t} - e^{-2t}}{2} \right]$$

$$\Rightarrow x_o(t) = -\sinh(2t) \cos t$$

2. (d)

$$x[n] = \cos\left(\frac{n}{4}\right) \sin\left(\frac{nx}{4}\right); \quad \frac{1}{4} = \frac{m}{N}$$

$$\Rightarrow \frac{m}{N} = \frac{1}{8\pi} \text{ which is irrational}$$

Hence, the signal is aperiodic.

3. (0.5)

$$x[n] = (-1)^n u[n]$$

Power, $P = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N x^2[n]$

$$P = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=0}^N (-1)^{2n} = \lim_{N \rightarrow \infty} \frac{N+1}{2N+1}$$

By using L-Hospital's rule, $P = \frac{1}{2} \text{ W} = 0.5 \text{ W}$

4. (3)

$$x[n] = \cos \frac{n\pi}{3} [u[n] - u[n-6]]$$

Energy, $E = \sum_{n=-\infty}^{\infty} x^2[n]$

$$E = \sum_{n=0}^5 \left[\cos \frac{n\pi}{3} \right]^2$$

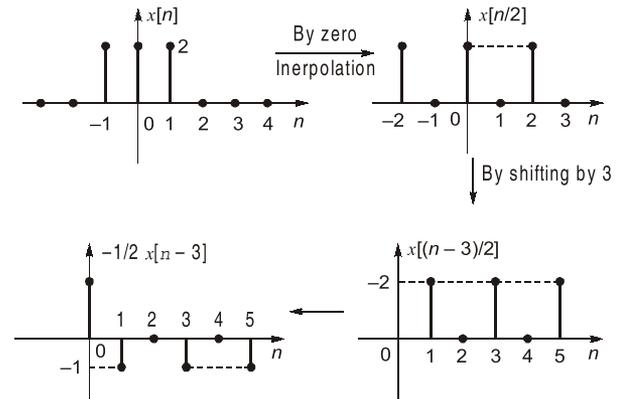
$$\Rightarrow E = [\cos 0]^2 + \left[\cos \frac{\pi}{3} \right]^2 + \left[\cos \frac{2\pi}{3} \right]^2 + [\cos \pi]^2$$

$$+ \left[\cos \left[\frac{4\pi}{3} \right] \right]^2 + \left[\cos \left[\frac{5\pi}{3} \right] \right]^2$$

$$E = 1 + \frac{1}{4} + \frac{1}{4} + 1 + \frac{1}{4} + \frac{1}{4}$$

$$E = 3 \text{ J}$$

5. (4.5)

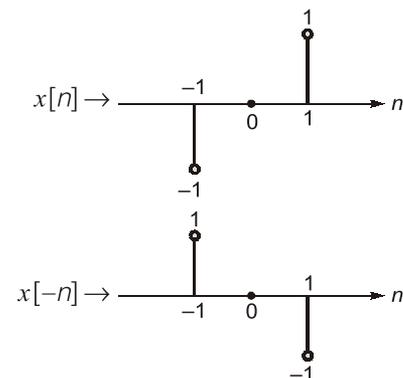


So, $y[n] = -\frac{1}{2} x \left[\frac{n-3}{2} \right] = \alpha x \left[\frac{n-n_o}{K} \right]$

$$\alpha = -\frac{1}{2}; \quad n_o = 3; \quad K = 2$$

So, $n_o + \alpha + K = 4.5$

6. (a)



$\therefore y[n] = x[n] + x[-n] = 0$

\Rightarrow Energy of $y[n] = \sum_{n=-\infty}^{\infty} |y(n)|^2 = 0$

7. (c)

$$x(t) = \underbrace{4 \cos \left(\frac{2\pi}{3} t + 40^\circ \right)}_{x_1(t)} + \underbrace{3 \sin \left(\frac{4\pi}{5} t + 20^\circ \right)}_{x_2(t)}$$

$$\omega_1 = \frac{2\pi}{3} \Rightarrow T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{2\pi/3} = 3$$



CONVENTIONAL BRAIN TEASERS

Q.1 Consider a signal $x[n]$,

$$x[n] = \sum_{k=-\infty}^{\infty} g[n-k] + \sum_{k=-\infty}^{\infty} g\left[\frac{n}{N} - k\right] \quad \text{where, } g[n] = \delta[n]$$

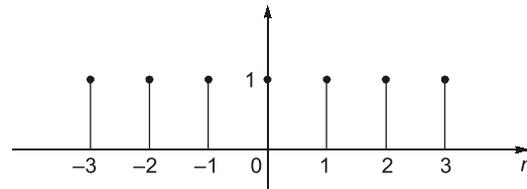
N is an even integer with $0 < N < 4$.

(i) Draw the waveform of signal $x[n]$. (ii) Find the power of the signal $x[n]$.

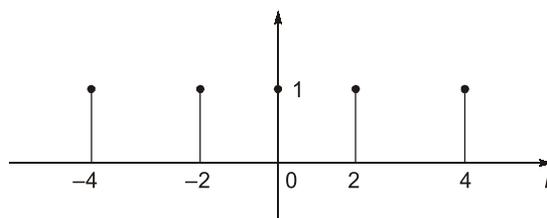
(iii) Find the value of $\sum_{n=-4}^5 x[n]$.

1. (Sol.)

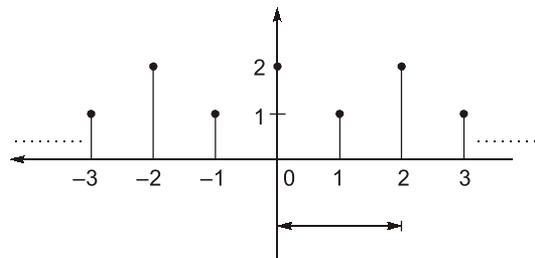
(i) $\sum_{k=-\infty}^{\infty} \delta[n-k]$



$$\sum_{k=-\infty}^{\infty} \delta\left[\frac{n}{2} - k\right]$$



$x[n] \Rightarrow$



$x[n]$ is periodic with period. $M = 2$.

(ii) Power of $x[n]$
$$P_x = \frac{1}{M} \sum_{n=0}^1 |x[n]|^2 = \frac{1}{2} [4 + 1] = \frac{5}{2} = 2.5 \text{ W}$$

(iii)
$$\sum_{n=-4}^5 x[n] = 2 + 1 + 2 + 1 + 2 + 1 + 2 + 1 + 2 + 1 = 15$$

